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An asymmetric theory of nonlocal elasticity—Part 2. Continuum field

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Abstract

In this paper, an asymmetric theory of nonlocal elasticity with nonlocal body couple is developed on the basis of the axiom system in nonlocal continuum field theory. The Galileo invariance is used for determining the explicit form of the constitutive equations. It is shown that both continuum field theory and quasicontinuum theory give the same constitutive equations and field equations for the general theory of nonlocal elasticity. Finally, the relations among nonlocal theory, couple stress theory, and higher gradient theory are investigated. © 1999 Published by Elsevier Science Ltd. All rights reserved.

1. Introduction

In a previous paper [Part 1 of asymmetry theory of nonlocal elasticity (Gao, 1999)], a more general model of nonlocal elasticity in quasicontinuum field theory has been developed. It has been shown that the local rotation of an atomic lattice plays a very important role in solid mechanics. Both symmetric stress and antisymmetric stress are nonlocal function of strain and local rotation. The nonlocal constitutive parameters are explicitly expressed in terms of the force constants connecting the atomic lattices.

This paper aims to develop a general theory of nonlocal elasticity according to nonlocal continuum field theory developed by Eringen (Eringen, 1976). First the constitutive functional is selected based on the dual relationship between strain and symmetric stress, local rotation and nonlocal body couple, respectively. It is also shown that the two restriction conditions correspondent to the rectilinear uniform motion and rigid body rotation for internal energy derived from the Galileo invariance are sufficient conditions of nonlocal conservative laws. The internal energy can be expressed in integral form due to Friedman and Katz's representative theorem. The kernel function of the internal energy in integral from (or called as local internal energy) is expanded as a polynomial function of strain and local rotation. Thus, the constitutive equations of symmetric stress and nonlocal body couple are explicitly expressed as the nonlocal function of strain and local rotation. Finally, the relations among nonlocal theory, couple stress theory and higher gradient theory are also investigated.

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2. The conservation laws of nonlocal theory

Since the characteristics of interacting force between atoms is long range, a body being in motion is regarded as a whole organic body. Thus it is assumed that the conservation laws in nonlocal field theory governing the mechanical state of a continuum body is integral forms. For small deformation, the conservation laws in integral form is localized by means of Green–Gauss theorem. The localizable conservation laws are given by:

$$\hat{\rho} = \rho + \rho V^m|_m \quad \int_v \hat{\rho}\,\mathrm{d}v = 0 \tag{1}$$

$$\rho\dot{\boldsymbol{V}} + \hat{\rho}\boldsymbol{V} - \rho\boldsymbol{f} - \boldsymbol{t}^k|_m = \rho\hat{\boldsymbol{F}} \quad \int_v \rho\hat{\boldsymbol{F}}\,\mathrm{d}v = 0 \tag{2}$$

$$\rho\boldsymbol{x}\times\hat{\boldsymbol{F}} - \boldsymbol{e}_k\times\boldsymbol{t}^k - \rho\boldsymbol{c} = \rho\hat{\boldsymbol{L}} \quad \int_v \rho\hat{\boldsymbol{L}}\,\mathrm{d}v = 0 \tag{3}$$

$$\rho U = \hat{\rho}\left(\frac{1}{2}\boldsymbol{V}\cdot\boldsymbol{V} - U\right) - \rho\hat{\boldsymbol{F}}\cdot\boldsymbol{V} + \boldsymbol{t}^k\cdot\boldsymbol{V}_{,k} + \frac{1}{2}(\rho\boldsymbol{x}\times\hat{\boldsymbol{F}} - \boldsymbol{e}_k\times\boldsymbol{t}^k - \rho\hat{\boldsymbol{L}})\cdot\nabla\times\boldsymbol{V} + \rho\hat{h} \quad \int_v \rho\hat{h}\,\mathrm{d}v = 0 \tag{4}$$

where ρ is the density of mass, U is the internal energy, $\boldsymbol{V}$ is the velocity vector, $\boldsymbol{x}$ is the position vector, $\boldsymbol{f}$ is the body force vector, $\boldsymbol{t}^k$ is the vector of surface force, $\boldsymbol{c}$ is the external body couple vector, $\boldsymbol{t}_n$ is the stress vector in the normal direction $\boldsymbol{n}$ (see Eringen, 1976; Gao and Chen, 1992).

$\hat{\rho}$, $\hat{\boldsymbol{F}}$, $\hat{\boldsymbol{L}}$, $\hat{h}$ are, respectively, nonlocal mass, nonlocal body force, nonlocal body couple and nonlocal energy. These nonlocal variables describe the global properties of a body and must satisfy the nonlocal conservative laws (see, Eringen, 1976). It is noted that when the chemical action in a body is ignored, $\hat{\rho} = 0$.

We decompose the work done by stresses into two parts: one is from symmetric stress and strain rate; another is from antisymmetric stress and local rotation rate, given by

$$\boldsymbol{t}^k\cdot\boldsymbol{V}_{,k} = t^s_{kl}\dot{e}_{kl} + t^a_{kl}\dot{\omega}_{lk} \tag{5}$$

where

$$\dot{e}_{kl} = \tfrac{1}{2}(V_{k,l} + V_{l,k}), \quad \dot{\omega}_{lk} = \tfrac{1}{2}(V_{k,l} - V_{l,k}); \quad \text{and} \quad t^s_{kl} = \tfrac{1}{2}(t_{kl} + t_{lk}), \quad t^a_{kl} = \tfrac{1}{2}(t_{kl} - t_{lk}).$$

When the external body couple is neglected, the antisymmetric stress is correspondent to nonlocal couple, i.e.

$$e_{ijk}\rho\hat{J}_k = t^a_{ij} \tag{6}$$

where $\hat{\boldsymbol{J}} = \boldsymbol{x}\times\hat{\boldsymbol{F}} - \hat{\boldsymbol{L}}$. In this case, the conservation law of energy becomes

$$\rho \dot{U} = \boldsymbol{t}^s \cdot \dot{\boldsymbol{e}} - \rho \hat{\boldsymbol{F}} \cdot \boldsymbol{V} + \rho \hat{\boldsymbol{J}} \cdot \dot{\theta} + \rho \hat{h} \tag{7}$$

where $\dot{\theta} = \frac{1}{2} \nabla \times \boldsymbol{V}$.

The local rotation can make a contribution to strain energy because of the nonlocal effect, but in classical mechanics, the local rotation is regarded as a rigid body rotation not to be associated with the deformation of a body. However, the local rotation is the relative rotation between atoms or particles and is a very important characteristic variable representing the deformation of a body (as discussed in Part 1; see Gao, 1999). In fact, the compatible conditions of small deformation are still represented by local rotation angle. For example, the compatible condition of geometric deformation

$$\frac{\partial^2 e_{11}}{\partial x_2^2} + \frac{\partial^2 e_{22}}{\partial x_1^2} = \frac{\partial^2 e_{12}}{\partial x_1 \partial x_2} \tag{8}$$

can be rewritten as

$$\frac{\partial}{\partial x_2}\left(\frac{\partial \theta_3}{\partial x_1}\right) = \frac{\partial}{\partial x_1}\left(\frac{\partial \theta_3}{\partial x_2}\right) \tag{9}$$

which is also the smoothing continuous condition of local rotation. If the equation is violated, the distribution of the local rotation in a deformation body is not continuous and a microcrack can be initiated. Also the eqn (7) shows that the dual variable of the local rotation is nonlocal body couple $\rho \hat{\boldsymbol{J}}$. In the general theory of nonlocal elasticity without micropolar rotation, the effect of local rotation should be considered and the stress should be asymmetric.

3. The conservation relation of nonlocal elasticity with nonlocal body couple

From the energy conservative law given by eqn (7), we select the constitutive variables as follows

$$\chi = [\boldsymbol{e}, \theta, \boldsymbol{x}]; \quad \chi' = [\boldsymbol{e}', \theta', \boldsymbol{x}'] \tag{10}$$

where $\boldsymbol{e}$ is a strain tensor; $\theta (= \nabla \times \boldsymbol{u})$ is a local rotation vector; a prime $()'$ placed on quantities indicates that they depend on $\boldsymbol{x}'$, e.g.

$$\boldsymbol{e}' = \boldsymbol{e}(\mathbf{x}', t), \quad \theta' = \theta(\boldsymbol{x}', t) \tag{11}$$

$\boldsymbol{x}'$ represents the position vector of any particle in the domain v occupied by a body.

According to the axiom of Causality in nonlocal theory, the mechanical state of a point depends on the motion of all particles of a body. It is assumed that the constitutive functional of the internal energy is

$$U = U(\chi, \chi') \tag{12}$$

By substituting the internal energy given by eqn (12) into eqn (7), we obtain a linear equation of χ. For all independent motions throughout v, the linear equation is true if and only if

$$-\rho\hat{\boldsymbol{F}} = \rho\frac{\partial U}{\partial \boldsymbol{x}} + \int_v \left(\rho\frac{\delta U}{\delta \boldsymbol{x}'}\right)^* \mathrm{d}v(\boldsymbol{x}')$$

$$\boldsymbol{t}^s = \rho\frac{\partial U}{\partial \boldsymbol{e}} + \int_v \left(\rho\frac{\delta U}{\delta \boldsymbol{e}'}\right)^* \mathrm{d}v(\boldsymbol{x}')$$

$$\rho\hat{\boldsymbol{J}} = \rho\frac{\partial U}{\partial \theta} + \int_v \left(\rho\frac{\delta U}{\delta \theta'}\right)^* \mathrm{d}v(\boldsymbol{x}')$$

$$\mathscr{L} - \rho\hat{h} = 0 \tag{13}$$

where

$$\mathscr{L} = \int_v \left[\left(\rho\frac{\delta U}{\delta \chi'}\cdot\dot{\chi}'\right) - \left(\rho\frac{\delta U}{\delta \chi'}\cdot\dot{\chi}(\boldsymbol{x}')\right)^*\right]\mathrm{d}v(\boldsymbol{x}')$$

which are the constitutive equations in nonlocal elasticity with the action of nonlocal body couple (the detailed calculation is seen in Gao and Chen, 1992).

The constitutive functional must satisfy Galilean invariance. Here, we consider the case that both the internal energy and rate of the internal energy are invariant with respect to the rigid motion of the body (see Mason, 1980). First, let us consider rectilinear uniform motion

$$\boldsymbol{x} \to \bar{\boldsymbol{x}} = \boldsymbol{x} + \boldsymbol{V}^0\boldsymbol{t}; \quad (\boldsymbol{V}^0 \text{ is an arbitrary constant vector}) \tag{14}$$

From the Galilean invariance, we obtain

$$\dot{U}(\boldsymbol{e}, \theta, \bar{\boldsymbol{x}}; \boldsymbol{e}', \theta', \bar{\boldsymbol{x}}') = \dot{U}(\boldsymbol{e}, \theta, \boldsymbol{x}; \boldsymbol{e}', \theta', \boldsymbol{x}') \tag{15}$$

By expanding the above equation and eliminating the same terms in both sides of the equation, we have the restriction condition of the constitutive function, given by

$$\rho\frac{\partial U}{\partial \boldsymbol{x}} + \int_v \left(\rho\frac{\delta U}{\delta \boldsymbol{x}'}\right)^* \mathrm{d}v(\boldsymbol{x}') + \int_v \left(\left(\rho\frac{\delta U}{\delta \boldsymbol{x}'}\right) - \left(\rho\frac{\delta U}{\delta \boldsymbol{x}'}\right)^*\right)\mathrm{d}v(\boldsymbol{x}') = 0 \tag{16}$$

The constitutive equation of nonlocal body force $\hat{\boldsymbol{F}}$ given by (13a) can be rewritten as follows

$$\rho\hat{\boldsymbol{F}} = \int_v \left(\left(\rho\frac{\delta U}{\delta \boldsymbol{x}'}\right) - \left(\rho\frac{\delta U}{\delta \boldsymbol{x}'}\right)^*\right)\mathrm{d}v(\boldsymbol{x}') \tag{17}$$

Since the integrand in eqn (17) is symmetric on the integral variables $\boldsymbol{x}'$ and $\boldsymbol{x}$, the conservation law on the nonlocal body force is satisfied.

Second, let us consider a rigid body rotation with a constant rotational velocity ω_0:

$$\dot{\boldsymbol{x}} \to \bar{\boldsymbol{x}} = \dot{\boldsymbol{x}} + \omega_0 \times \boldsymbol{x}$$

$$\boldsymbol{x} \to \bar{\boldsymbol{x}} = \boldsymbol{x} + \phi; \quad \frac{\mathrm{d}\phi}{\mathrm{d}t} = \omega_0 \times \boldsymbol{x} \tag{18}$$

For Galilean invariance, we have

$$\frac{\partial U}{\partial \chi}+\int_v \frac{\partial U}{\partial \chi'}\mathrm{d}v(\boldsymbol{x}') = \frac{\partial U}{\partial \bar{\chi}}+\int_v \frac{\partial U}{\partial \bar{\chi}'}\mathrm{d}v(\boldsymbol{x}')$$

$$\dot{U}[\boldsymbol{e},\bar{\theta},\bar{\boldsymbol{x}};\boldsymbol{e}',\bar{\theta}',\bar{\boldsymbol{x}}'] = \dot{U}[\boldsymbol{e},\theta,\boldsymbol{x};\boldsymbol{e}',\theta',\boldsymbol{x}'] \tag{19}$$

where $\bar{\chi} = [\boldsymbol{e},\bar{\theta},\bar{\boldsymbol{x}}]$; $\bar{\chi}' = [\boldsymbol{e}',\bar{\theta}',\bar{\boldsymbol{x}}']$.

By expanding eqn (19b) and eliminating the same terms on both sides, and from the property of the vector analysis, we have

$$\boldsymbol{x}\times\left[\rho\frac{\partial U}{\partial \boldsymbol{x}}+\int_v\left(\rho\frac{\delta U}{\delta \boldsymbol{x}'}\right)^*\mathrm{d}v(\boldsymbol{x}')+\rho\frac{\partial U}{\partial \theta}+\int_v\left(\rho\frac{\delta U}{\delta \theta'}\right)^*\mathrm{d}v(\boldsymbol{x}')\right]$$
$$= -\int_v\left[\boldsymbol{x}'\times\left(\rho\frac{\delta U}{\partial \boldsymbol{x}'}\right)-\boldsymbol{x}\times\left(\rho\frac{\delta U}{\delta \boldsymbol{x}'}\right)^*+\left(\rho\frac{\delta U}{\delta \theta'}\right)-\left(\rho\frac{\delta U}{\delta \theta'}\right)^*\right]\mathrm{d}v(\boldsymbol{x}') \tag{20}$$

which is another restriction condition on the constitutive function U from the Galileo invariance. Thus, the constitutive equation of nonlocal body couple $\hat{\boldsymbol{L}}$ can be rewritten as follows

$$\begin{aligned}\rho\hat{\boldsymbol{L}} &= \rho(\boldsymbol{x}\times\hat{\boldsymbol{F}}-\hat{\boldsymbol{J}})\\ &= -\boldsymbol{x}\times\left[\rho\frac{\partial U}{\partial \boldsymbol{x}}+\int_v\left(\rho\frac{\delta U}{\delta \boldsymbol{x}'}\right)^*\mathrm{d}v(\boldsymbol{x}')-\rho\frac{\partial U}{\partial \theta}+\int_v\left(\rho\frac{\delta U}{\delta \theta'}\right)^*\mathrm{d}v(\boldsymbol{x}')\right]\\ &= \int_v\left[\boldsymbol{x}'\times\left(\rho\frac{\delta U}{\partial \boldsymbol{x}'}\right)-\boldsymbol{x}\times\left(\rho\frac{\delta U}{\delta \boldsymbol{x}'}\right)^*+\left(\rho\frac{\delta U}{\delta \theta'}\right)-\left(\rho\frac{\delta U}{\delta \theta'}\right)^*\right]\mathrm{d}v(\boldsymbol{x}')\end{aligned} \tag{21}$$

Therefore, the conservation law for nonlocal body couple $\int_v \rho\boldsymbol{L}\,\mathrm{d}v = 0$ is satisfied since the integrand is symmetric on $\boldsymbol{x}'$ and $\boldsymbol{x}$.

From the symmetry of the integral function $\mathscr{L}$ on $\boldsymbol{x}'$ and $\boldsymbol{x}$ given in eqn (13), we have

$$\int_v \rho\hat{h}\,\mathrm{d}v = \int_v \mathscr{L}\,\mathrm{d}v = 0 \tag{22}$$

Thus, we obtain the theorem that the sufficient condition of nonlocal physical quantities $(\hat{\boldsymbol{F}},\hat{\boldsymbol{L}})$ satisfying the nonlocal conservation laws is that the constitutive functional satisfies the Galilean invariance, i.e. the internal energy must satisfy the restriction eqns (16) and (20).

4. The linear theory

According to the discussion given by Eringen (1981, 1983), it is not necessary to employ a general functional for describing nonlocal behavior of most materials. In the sense of Friedman and Katz (1966), the additive functional is adequate to describe behavior of nonlocal solids. From the representative theorem of additive functional proposed by Friedman and Katz, we have

$$\rho U = \int_v \Psi[\boldsymbol{x}, \boldsymbol{x}'; \theta, \theta'; \boldsymbol{e}, \boldsymbol{e}']\, \mathrm{d}v(\boldsymbol{x}') \tag{23}$$

In this case, Frechet derivative for the constitutive functional U can be calculated by

$$\int_v \frac{\delta(\rho U)}{\delta \boldsymbol{G}'} \mathrm{d}v(\boldsymbol{x}') = \int_v \frac{\partial \psi}{\partial \boldsymbol{G}'} \mathrm{d}v(\boldsymbol{x}') \tag{24}$$

where $\boldsymbol{G}'$ represents any of $\boldsymbol{e}'$, θ' and $\boldsymbol{x}'$.

Substituting eqn (23) into eqn (16), we have the restriction condition on ψ, given by

$$\int_v \left(\frac{\partial \psi}{\partial \boldsymbol{x}} + \frac{\partial \psi}{\partial \boldsymbol{x}'} \right) \mathrm{d}v(\boldsymbol{x}') = 0 \tag{25}$$

For homogeneous materials, eqn (25) is not violated if and only if

$$\psi(\boldsymbol{x}, \boldsymbol{x}'; \theta, \theta'; \boldsymbol{e}, \boldsymbol{e}') = \psi(|\boldsymbol{x} - \boldsymbol{x}'|; \theta, \theta'; \boldsymbol{e}, \boldsymbol{e}') \tag{26}$$

Substituting eqn (26) into eqn (20), we have another restricted condition given by

$$\int_v \left(\frac{\partial \psi}{\partial \theta} + \frac{\partial \psi}{\partial \theta'} \right) \mathrm{d}v(\boldsymbol{x}') = 0 \tag{27}$$

By expanding the function ψ as polynomial function and eliminating the higher order terms, we have

$$\psi(|\boldsymbol{x} - \boldsymbol{x}'|; \theta, \theta'; \boldsymbol{e}, \boldsymbol{e}') = \psi_\theta(|\boldsymbol{x} - \boldsymbol{x}'|; \theta, \theta') + \psi_e(|\boldsymbol{x} - \boldsymbol{x}'|; \boldsymbol{e}, \boldsymbol{e}') + \psi_{\theta e}(|\boldsymbol{x} - \boldsymbol{x}'|; \theta, \theta'; \boldsymbol{e}, \boldsymbol{e}') \tag{28}$$

where

$$\psi_\theta(|\boldsymbol{x} - \boldsymbol{x}'|; \theta, \theta') = \psi_0(|\boldsymbol{x} - \boldsymbol{x}'|) \colon \theta' \otimes \theta' + \psi_1(|\boldsymbol{x} - \boldsymbol{x}'|) \colon \theta \otimes \theta' + \psi_2(|\boldsymbol{x} - \boldsymbol{x}'|) \colon \theta \otimes \theta$$

$$\psi_e(|\boldsymbol{x} - \boldsymbol{x}'|; \boldsymbol{e}, \boldsymbol{e}') = \psi_3(|\boldsymbol{x} - \boldsymbol{x}'|) \colon \boldsymbol{e}' \otimes \boldsymbol{e}' + \psi_4(|\boldsymbol{x} - \boldsymbol{x}'|) \colon \boldsymbol{e} \otimes \boldsymbol{e}' + \psi_5(|\boldsymbol{x} - \boldsymbol{x}'|) \colon \boldsymbol{e} \otimes \boldsymbol{e}$$

$$\psi_{\theta e}(|\boldsymbol{x} - \boldsymbol{x}'|; \theta, \theta'; \boldsymbol{e}, \boldsymbol{e}') = \psi_6(|\boldsymbol{x} - \boldsymbol{x}'|) \colon \boldsymbol{e} \otimes \theta + \psi_7(|\boldsymbol{x} - \boldsymbol{x}'|) \colon \boldsymbol{e} \otimes \theta' + \psi_8(|\boldsymbol{x} - \boldsymbol{x}'|) \colon \boldsymbol{e}' \otimes \theta + \psi_9(|\boldsymbol{x} - \boldsymbol{x}'|) \colon \boldsymbol{e}' \otimes \theta'$$

Since θ, θ'; $\boldsymbol{e}$, $\boldsymbol{e}'$ can be regarded as independent variables, substituting eqn (28) into eqn (27) leads to the restriction condition for the functions ψ_i ($i = 0, 1, 2, 6, 7, 8, 9$), given by

$$\psi_0 = \psi_2 = -\tfrac{1}{2}\psi_1$$

$$\psi_6 + \psi_7 = 0; \quad \psi_8 + \psi_9 = 0 \tag{29}$$

Therefore, we obtain

$$\psi_\theta(|\boldsymbol{x} - \boldsymbol{x}'|, \theta, \theta') = \psi_0(|\boldsymbol{x} - \boldsymbol{x}'|) \colon \Theta' \otimes \Theta'$$

$$\psi_{\theta e}(|\boldsymbol{x} - \boldsymbol{x}'|; \theta, \theta'; \boldsymbol{e}, \boldsymbol{e}') = \psi_7(|\boldsymbol{x} - \boldsymbol{x}'|) \colon \boldsymbol{e} \otimes \Theta' + \psi_9(|\boldsymbol{x} - \boldsymbol{x}'|) \colon \boldsymbol{e}' \otimes \Theta' \tag{30}$$

where $\Theta' = \theta' - \theta$.

It is noted that $\psi_0 = \psi_0^*$. We assume that $\psi_e = \psi_e^*$. Substituting ψ_0, ψ_{0e} given by eqn (30) and ψ_e given in eqn (28) into eqn (17) and eqn (21), respectively, we obtain $\rho\hat{\boldsymbol{F}} = 0$ and

$$\rho\hat{\boldsymbol{L}} = -\rho\hat{\boldsymbol{J}}\int_v\left[\frac{\partial\psi}{\partial\theta'} - \left(\frac{\partial\psi}{\partial\theta'}\right)^*\right]\mathrm{d}v(\boldsymbol{x}') = 4\int_v \psi_0(|\boldsymbol{x}-\boldsymbol{x}'|)\cdot\Theta'\,\mathrm{d}v(\boldsymbol{x}') + 2\Psi_6\colon\boldsymbol{e} + 2\int_v \psi_8(|\boldsymbol{x}-\boldsymbol{x}'|)\colon\boldsymbol{e}\,\mathrm{d}v(\boldsymbol{x}') \tag{31}$$

where

$$\Psi_6 = \int_v \psi_6(|\boldsymbol{x}-\boldsymbol{x}'|)\,\mathrm{d}v(\boldsymbol{x}')$$

By substituting eqn (28) into eqn (13b), we have

$$\boldsymbol{t}^s = \int_v\left[\frac{\partial\psi_e}{\partial\boldsymbol{e}} + \left(\frac{\partial\psi_e}{\partial\boldsymbol{e}'}\right)^*\right]\mathrm{d}v(\boldsymbol{x}') = \Sigma_1\colon\boldsymbol{e} + \int_v \Sigma_2(|\boldsymbol{x}-\boldsymbol{r}'|)\colon\boldsymbol{e}'\,\mathrm{d}v(\boldsymbol{x}') + \int_v [\psi_7(|\boldsymbol{x}-\boldsymbol{x}'|) + \psi_8(|\boldsymbol{x}-\boldsymbol{x}'|)]\colon\Theta'\,\mathrm{d}v(\boldsymbol{x}') \tag{32}$$

where

$$\Sigma_1 = 4\int_v \psi_3(|\boldsymbol{x}-\boldsymbol{x}'|)\,\mathrm{d}v(\boldsymbol{x}');\quad \Sigma_2(|\boldsymbol{x}-\boldsymbol{x}'|) = 2\psi_4(|\boldsymbol{x}-\boldsymbol{x}'|)$$

For isotropic, nonlocal and elastic materials, the tensors $(\psi_0, \Sigma_1, \Sigma_2)$; ψ_i $(i = 6, 7, 8, 9)$ are isotropic. It is noted that ψ_i $(i = 6, 7, 8, 9)$ are isotropic tensors of third rank. The isotropic tensors of third rank are expressed as $\psi_i = \psi_i\boldsymbol{\varepsilon}$ (ψ_i $(i = 6, 7, 8, 9)$ are scalars; $\boldsymbol{\varepsilon}$ is Eddington tensor). From the constitutive equations of symmetric stress and nonlocal body couple, the isotropic tensors ψ_i $(i = 6, 7, 8, 9)$ should be symmetric with respect to two of the subscripts. Thus, ψ_i $(i = 6, 7, 8, 9) = 0$. Other isotropic tensors are expressed as follows

$$\begin{aligned}
4\psi_0(|\boldsymbol{x}-\boldsymbol{x}'|) &= C_0(|\boldsymbol{x}-\boldsymbol{x}'|)\delta_{kl}\boldsymbol{e}_k\otimes\boldsymbol{e}_l\\
\Sigma_1 &= [\lambda_0\delta_{ij}\delta_{kl} + \mu_0\delta_{ik}\delta_{jl} + v_0\delta_{il}\delta_{jk}](\boldsymbol{e}_i\otimes\boldsymbol{e}_k\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_l)\\
\Sigma_2(|\boldsymbol{x}-\boldsymbol{x}'|) &= [\lambda_1(|\boldsymbol{x}-\boldsymbol{x}'|)\delta_{ij}\delta_{kl} + \mu_1(|\boldsymbol{x}-\boldsymbol{x}'|)\delta_{ik}\delta_{jl}\\
&\quad + v_1(|\boldsymbol{x}-\boldsymbol{x}'|)\delta_{il}\delta_{jk}](\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k\otimes\boldsymbol{e}_l)\\
\varepsilon &= \boldsymbol{e}_{ijk}\boldsymbol{e}_k\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k
\end{aligned} \tag{33}$$

where $\boldsymbol{e}_k$ $(k = 1, 2, 3)$ are base vectors of Cartesian coordinate system.

Then, the developed constitutive equations of nonlocal symmetric stress and nonlocal body couple are, respectively, nonlocal function of the strain and local rotation, given by

$$\rho\hat{\boldsymbol{L}} = \int_v C_0(|\boldsymbol{x}-\boldsymbol{x}'|)\Theta(\boldsymbol{x}')\,\mathrm{d}v(\boldsymbol{x}')$$

$$\boldsymbol{t}^s = \lambda_0 \boldsymbol{I}\,\mathrm{tr}\,\boldsymbol{e} + (\mu_0+v_0)\boldsymbol{e} + \int_v [\lambda_1(|\boldsymbol{x}-\boldsymbol{x}'|)\boldsymbol{I}\,\mathrm{tr}\,\boldsymbol{e}' + (\mu_1(|\boldsymbol{x}-\boldsymbol{x}'|)+v_1(|\boldsymbol{x}-\boldsymbol{x}'|))\boldsymbol{e}']\,\mathrm{d}v(\boldsymbol{x}') \tag{34}$$

where $\boldsymbol{I} = \delta_{ij}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j)$.

As discussed in Part 1 (Gao, 1999), the nonlocal characteristic functions C_0, λ_1, μ_1 and v_1 are, respectively, moment and force constants connecting atomic lattices. For homogeneous, isotropic solids, the field equations of the theory are

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \rho f_i + t_{ji,j}$$

$$e_{ijk}t_{ij} + \rho\hat{L}_k = 0$$

$$t_{ij} = t^s_{ij} + t^a_{ij} \tag{35}$$

Thus, the general theory of nonlocal elasticity with nonlocal body couple has been developed. It is noted that it is an asymmetric theory and is the same as that developed from quasicontinuum field theory. For isotropic materials, the symmetric stress and nonlocal body couple given in eqn (33) are nonlocal function of strain and local rotation, respectively. For anisotropic materials, both symmetric stress and nonlocal body couple given in eqns (31) and (32) are nonlocal function of strain and local rotation. The couple effects of local rotation on symmetric stress or strain on nonlocal body couple can be found out in the localized deformation in metal and deformed composite, such as kink band, etc. A further discussion for anisotropic materials will be in another paper. In the following the discussion of asymmetric theory of nonlocal elasticity focuses on the isotropic material.

5. Discussion

5.1. The nonlocal property of asymmetric stress

The nonlocal body couple exists in the nonlocal media. From the equilibrium equation of moments given by eqn (35) and constitutive equation of nonlocal body couple, we have

$$\boldsymbol{e}\colon\quad \boldsymbol{t} = -\rho\hat{\boldsymbol{L}} = -\int_v C_0(|\boldsymbol{x}-\boldsymbol{x}'|)\Theta(\boldsymbol{x}')\,\mathrm{d}v(\boldsymbol{x}') \tag{36}$$

which indicates that the nonlocal body couple is caused by nonlocal effect of local rotation. When the nonlocal effect of local rotation is neglected, i.e. $\rho\hat{\boldsymbol{L}} = 0$, the antisymmetric stress disappears. The theory can be reduced to the Kroner–Eringen model of the nonlocal elasticity. In this case, due to the definition of nonlocal functional space, the nonlocal characteristic function $C_0(x)$ has the following characteristics

(i) $|\boldsymbol{x}-\boldsymbol{x}'| \to \infty; \quad C_0(|\boldsymbol{x}-\boldsymbol{x}'|) \to 0;$

(ii) $\int_v C_0(|\boldsymbol{x}-\boldsymbol{x}'|)\,\mathrm{d}v = \text{constant};$

(iii) when nonlocal characteristic length $a \to 0, \quad C_0(|\boldsymbol{x}-\boldsymbol{x}'|) \to 0$

The nonlocal characteristic function $C_0(x)$ can be expressed explicitly as function of moment constants associated with atomic lattices inter-connecting and the interval distance of atoms regarded as the internal length (or nonlocal characteristic length) (see Gao, 1999). Two characteristic functions presented in Part 1 (Gao, 1999), the quasicontinuum function of discrete moments and a linear approximating function associated with nonlinear dispersion of plane wave of one-dimensional atomic lattice chain, satisfy the above requirement for nonlocal characteristic function. For isotropic material, the nonlocal effect leading to asymmetry comes from the long range property of interaction of atoms in the metal materials and non-uniform distribution of atomic forces in the more microscopic structures.

It should be indicated that the effect of $C_0(x)$ on the nonlinear dispersion of plane waves can be found out from the difference between longitudinal wave and transverse wave. The detailed and more discussion on $C_0(x)$ is seen in the paper (Gao and Chen, 1992).

5.2. *The couple stress theory is a special case of nonlocal asymmetric theory*

By expanding rotation angle $\theta(\boldsymbol{x}')$ at the point $\boldsymbol{x}$, we have

$$\theta(\boldsymbol{x}') = \theta(\boldsymbol{x}) + (\boldsymbol{x}'-\boldsymbol{x})\cdot\nabla\theta + \tfrac{1}{2}[(\boldsymbol{x}'-\boldsymbol{x})\cdot\nabla]^2\theta + \cdots \tag{37}$$

Substituting eqn (37) into eqn (36) and eliminating the higher gradient terms, we have

$$\rho\hat{\boldsymbol{L}} = \boldsymbol{R}\cdot\nabla\theta + \frac{\beta}{2}\nabla^2\theta \tag{38}$$

where

$$\boldsymbol{R} = \int_v (\boldsymbol{x}'-\boldsymbol{x})\cdot C_0(|\boldsymbol{x}-\boldsymbol{x}'|)\,\mathrm{d}v(\boldsymbol{x}')$$

$$\beta = \int_v [(\boldsymbol{x}'-\boldsymbol{x})\otimes(\boldsymbol{x}'-\boldsymbol{x})]C_0(|\boldsymbol{x}-\boldsymbol{x}'|)\,\mathrm{d}v(\boldsymbol{x}')$$

The nonlocal body couple on the boundary surface is mainly proportional to the first order gradient of the local rotation. It is more reasonable to consider the effect of first order gradient of local rotation on the Cosserat surface. If the nonlocal effect from the nearest atoms is only considered and due to the property of nonlocal characteristic functions suggested in Part 1, the nonlocal body couple inside the body is only proportional to the second gradient of the local rotation because $\boldsymbol{R} = 0$ inside the body. In this case that the surface nonlocal couple $\boldsymbol{R}$ is ignored, the antisymmetric stress can be expressed as the following

$$\boldsymbol{e}: \quad \boldsymbol{t} = -\frac{\beta}{2}\nabla^2\theta = -\frac{\beta}{4}\nabla^2(\nabla\times\boldsymbol{u}) \tag{39}$$

or

$$e_{ijk}t_{ij} + c_a\nabla^2\theta_k = 0 \tag{40}$$

where

$$\theta_k = -\frac{1}{2}e_{ijk}u_{[i,j]}, \quad c_a = \frac{\beta}{2}.$$

The equation is exactly the same as the equilibrium equation of moment in couple stress theory given by

$$e_{ijk}t_{ij} + M_{ik,i} = 0 \tag{41}$$

While the constitutive equation of the moment is

$$M_{ij} = c_a\frac{\partial\theta_j}{\partial x_i} \tag{42}$$

Substituting eqn (42) into eqn (41) reduces to eqn (40) regarded as the constitutive equation of antisymmetric stresses in nonlocal elasticity. It is noted that c_a is proportional to a^2C_0 (a is the interval distance of atoms) while C_0 is the physical constant associated with moment constants connecting the atomic lattices. The physical meaning of c_a is the modulus of moment stress.

We consider the case that the discrete system of Born's model is reduced by a granular material with contact of inter-particles, in which G_n and G_s represent the stiffnesses of contact moment along normal and tangential directions, respectively. From the micromechanics of granular material, we have that $c_a = \alpha(G_n - G_s)r^2$ (r is the radius of particle size and α is a geometric parameter) (see Chang and Gao, 1995). By comparing c_a with C_0 and contact stiffnesses G_n, G_s we also find out the physical meaning of C_0 which represents microscopic characteristics of the materials.

5.3. Higher gradient theory can be regarded as first order approximation of nonlocal theory

The constitutive equation of symmetric stress derived in this paper is the same as Kroner and Eringen's original work in nonlocal elasticity (Kroner, 1967; Eringen, 1972). The nonlocal characteristic functions proposed by Eringen (see Eringen, 1976; Eringen et al., 1977) are

$$[\lambda_1(|\boldsymbol{x}-\boldsymbol{x}'|), \quad \mu_1(|\boldsymbol{x}-\boldsymbol{x}'|)+v_1(|\boldsymbol{x}-\boldsymbol{x}'|)] = [\lambda_0, 2\mu_0]\alpha(|\boldsymbol{x}-\boldsymbol{x}'|) - [\lambda_0, 2\mu_0]\delta(|\boldsymbol{x}-\boldsymbol{x}'|) \tag{43}$$

where $[\lambda_0, \mu_0]$ are Lamé constants.

Then, the constitutive equation of symmetric stress becomes

$$t^s_{ij} = \int_v \alpha(|\boldsymbol{x}-\boldsymbol{x}'|)t^0_{ij}(\boldsymbol{x}')\,\mathrm{d}v(\boldsymbol{x}')$$

$$t^0_{ij} = \lambda_0\delta_{ij}e_{kk} + 2\mu_0 e_{ij} \tag{44}$$

which has the same form derived from the quasicontinuum field theory in Part 1 (Gao, 1999).

The nonlocal attenuating function $\alpha(|\boldsymbol{x}-\boldsymbol{x}'|)$ is a function of force constants connecting atoms and is determined from nonlinear dispersion relation in one dimension of atomic chain from the Born–Karman model (see Eringen, 1972). For the three-dimension case, the nonlocal attenuating function $\alpha(|\boldsymbol{x}|)$ with only interaction of the nearest atoms has been suggested to be (Chang and Gao, 1995):

$$\alpha(|\boldsymbol{x}|) = \frac{3}{\pi a^4}(a-|\boldsymbol{x}|); \quad \text{when } |\boldsymbol{x}| < a$$
$$= 0; \quad \text{when } |\boldsymbol{x}| > a \tag{45}$$

By expanding the integral function $t_{ij}^0(\boldsymbol{x}')$ at the point $\boldsymbol{x}$, i.e.

$$t_{ij}^0(\boldsymbol{x}') = t_{ij}^0(\boldsymbol{x}) + (x_k'-x_k)\frac{\partial}{\partial x_k}t_{ij}^0(\boldsymbol{x}) + \frac{1}{2}(x_k'-x_k)(x_l'-x_l)\frac{\partial^2}{\partial x_k\,\partial x_l}t_{ij}^0(\boldsymbol{x}) + \cdots \tag{46}$$

By substituting eqn (46) into eqn (44), eliminating higher terms and directly calculating the integrals, we get

$$t_{ij}^s = (1+c\nabla^2)(\lambda_0\delta_{ij}e_{kk}+2\mu_0 e_{ij}) \tag{47}$$

where

$$c = \frac{1}{2}\int_v \alpha(|\boldsymbol{x}-\boldsymbol{x}'|)(x_k'-x_k)(x_k'-x_k)\,\mathrm{d}v(\boldsymbol{x}').$$

If the antisymmetric part of the stress is neglected, the stress becomes symmetric. Then the nonlocal stress can be approximately expressed as the constitutive model with higher gradient of strain, given by

$$t_{ij} = t_{ij}^s = (1+c\nabla^2)(\lambda_0\delta_{ij}e_{kk}+2\mu_0 e_{ij}) \tag{48}$$

When $c < 0$, the constitutive equation reduces to that suggested by Beran and McCoy (1970). In fact, the constitutive modules c of higher gradient term is associated with the internal length of nonlocal media being the distance of atoms or lattice parameter. So, c should be positive. If we select the nonlocal characteristic function given in eqn (45), the higher gradient constant c is $a^2/10$. For the longitudinal wave of one-dimension continuum media, the displacement field is $\boldsymbol{u} = [u(x,t), 0, 0]$. The dynamic equation is that

$$\rho\ddot{u} = (\lambda_0+2\mu_0)\left(1+c\frac{\partial^2}{\partial x^2}\right)\frac{\partial^2 u}{\partial x^2} \tag{49}$$

The general solution for a single harmonic wave propagating along the direction x is that

$$u(x,t) = A\exp[i(kx-\omega t)] \tag{50}$$

Substituting eqn (50) into eqn (49) gives the dispersion relation

$$\omega = \omega_0 k\sqrt{1-ck^2} \tag{51}$$

where $\omega_0 = [\sqrt{(\lambda_0+2\mu_0)/\rho}]$; ω is frequency of wave; k is the wave number.

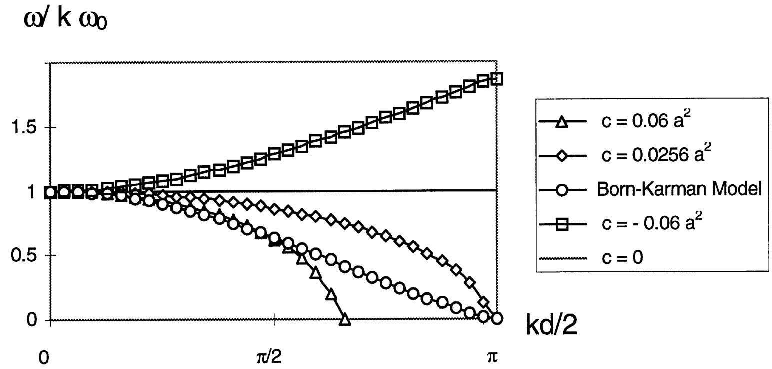

Fig. 1.

The dispersion relation given by the Born–Karman model (Born and Huang, 1954), in which the atoms are vibrating in a small region of their equilibrium position, gives

$$\omega = \omega_0 \left(\frac{2}{a}\right) \sin\left(k\frac{a}{2}\right) \quad (52)$$

where a is the lattice parameter.

As shown in Fig. 1, the parameter c in the linear theory of higher gradient elasticity must be positive and the best match is $\sqrt{c} = 0.245a$ (or $c = 0.06a^2$) in the Brillouin zone $[0 < k < (\pi/a)]$. Obviously, the higher gradient model with $c < 0$ is not proper for the material with atomic microstructure.

The asymmetric theory of nonlocal elasticity suggests the general model of higher gradient elasticity regarded as first-order approximation of the asymmetric nonlocal theory given by

$$t_{ij} = (1 + c\nabla^2)[\lambda_0 \delta_{ij} e_{kk} + 2\mu_0 e_{ij}] + \frac{c_a}{2}\nabla^2 u_{[i,j]} \quad (53)$$

Here the nonlocal effect on the surface of a body has been neglected; if $c = 0$, the model reduces to the couple stress theory. If $c = 0$ and $c_a = 0$, the model reduces to classic elasticity.

6. Conclusions

The general model of nonlocal elasticity developed on the basis of micro-behavior of crystal lattice and continuum field theory is asymmetric. The antisymmetric stress is caused by nonlocal effect of local rotation and anisotropy. The higher gradient model can be reduced from the nonlocal theory. The couple stress theory is a special case of higher gradient theory.

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